Dynamic Interaction Fields in a Two-Dimensional Lattice*

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Summary—**In the theory of artificial dielectrics and aperture coupling in rectangular waveguides, a knowledge of the dynamic interaction fields is required in order to evaluate the polarizing fields. This paper presents suitable methods for evaluating the dynamic interaction fields in a two-dimensional lattice. Both electric and magnetic dipoles are considered. The results are presented in closed form apart from correction terms involving rapidly converging series. Cross-polarization interaction constants are also evaluated.**

Introduction

TWO-DIMENSIONAL periodic lattice structures are encountered in the field of artificial dielectric media, aperture coupling in rectangular waveguides, and elsewhere. Fig. 1(a) illustrates a disk-type artificial dielectric. Each plane array of disks may be represented as an equivalent shunt susceptance B which loads a transmission line periodically along the z direction. Fig. 1(b) illustrates an aperture in a transverse wall in a rectangular guide, together with the images of the aperture in the guide walls. The shunt susceptance of planar arrays of the above type disks or apertures is usually determined by using the small aperture theory of Bethe (or its dual for the obstacle problem).[1] In this theory, each obstacle or aperture is replaced by an equivalent set of electric and magnetic dipoles, with moments given by the product of the incident field and a suitable polarizability constant which is dependent on the obstacle geometry only.[2] A limitation of the simple theory is that the obstacle size must be small and the spacing must be large, so that interaction between neighboring elements can be neglected.

In practice, it is usually desirable to employ such element spacings that the mutual interaction cannot be neglected. In such cases, the effective polarizing fields are the sum of the incident field and the field radiated by the induced dipoles in all the neighboring elements. For sufficiently small element spacing, the interaction field may be approximated by a static field. The evaluation of the interaction constant for a static interaction field has been carried out by Brown.[3] In this paper, suitable methods for evaluating the dynamic interaction field in closed form will be presented, and a restricted problem will be analyzed in detail. However, it should be noted that the methods of summing the series involved may be applied without difficulty to a general two-dimensional lattice with a field incident at any arbitrary angle.

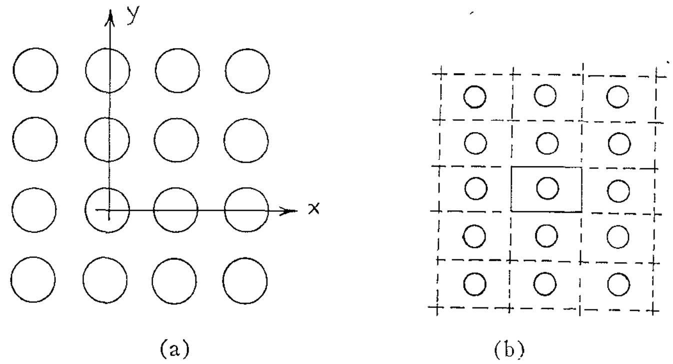

Fig. 1—Periodic arrays of similar elements.

An Array of Circular Disks

Fig. 2 illustrates a two-dimensional array of circular conducting disks such as is encountered in the field of artificial dielectric media. The spacing between disks is a along the x axis and b along the y axis. A perpendicular polarized TEM wave is assumed incident at an angle θ_i, relative to the z axis, in the xz plane. The incident field is

$$E_{\rm inc} = E_y = e^{-jhx-\Gamma_0 z}, \tag{1a}$$

$$H_{\rm inc} = H_z = \frac{h}{k_0} Y_0 E_{\rm inc}, \tag{1b}$$

where $h=k_0 \sin \theta_i$, $\Gamma_0=jk_0 \cos \theta_i$, $Y_0=(\epsilon_0/\mu_0)^{1/2}$. In (1) the x component of the magnetic field has not been written down. This incident field induces y-directed electric dipoles of moment P in each disk, as well as z-directed magnetic dipoles of moment M in each disk. In view of the nature of the incident field, the induced moment in the disk at $x=ma$ has a phase e^{-jhma} relative to the dipole located at $x=0$. The effective fields acting to polarize each disk are the sum of the incident fields plus the interaction field due to the fields radiated (scattered) by all of the neighboring disks. The interaction fields are proportional to the dipole strengths, and hence also proportional to the amplitude of the incident field.

The y-directed dipole moment P of each neighboring disk produces a y-directed electric interaction field E_{ie}

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[1] H. A. Bethe, "Theory of diffraction by small holes," *Phys. Rev.*, vol. 66, pp. 163–182; February, 1944.
[2] A. A. Oliner, "Equivalent circuits for small symmetrical longitudinal apertures and obstacles," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 72–80; January, 1960.
[3] J. Brown and W. Jackson, "The relative permittivity of tetragonal arrays of perfectly conducting thin disks," *Proc. IEE*, vol. 102, Pt. B, pp. 37–42; January, 1955.

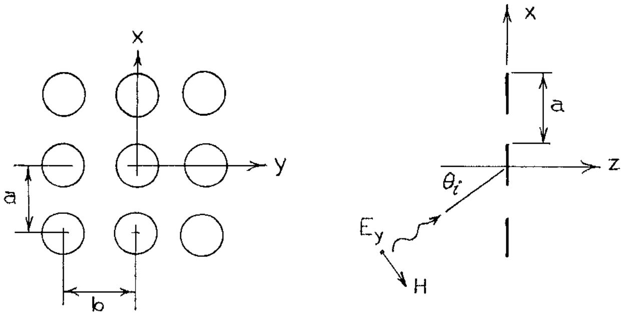

Fig. 2—Two-dimensional array of circular disks.

and a z-directed magnetic interaction field H_{ie} at the center of the disk, at the origin. Similarly, a z-directed magnetic dipole M in each neighboring disk produces a z-directed magnetic interaction field H_{im} and a y-directed electric interaction field E_{im} at the center of the disk at the origin. In addition, x- and y-directed magnetic interaction fields and x- and z-directed electric interaction fields are produced in general. This results in additional dipole moments in each obstacle. These cross-polarized dipole moments are small, however, since they are produced by the interaction fields only and do not have a contribution from the incident field. In most practical cases, these additional dipole moments may be neglected. The fact that they are present to some extent shows that even an array of isotropic particles in a cubical lattice structure will have anisotropic properties (structural anisotropy).[4]

In view of the linear relationship between the quantities involved, it is possible to write

$$E_i = E_{ie} + E_{im} = C_{ee}\frac{P}{\epsilon_0} + C_{em}Z_0M, \tag{2a}$$

$$H_i = H_{ie} + H_{im} = C_{me}Y_0\frac{P}{\epsilon_0} + C_{mm}M, \tag{2b}$$

where the interaction constants C_{ee}, C_{em}, C_{me} and C_{mm} are constants defined by these equations. The intrinsic impedance $Z_0=(\mu_0/\epsilon_0)^{1/2}$ and its reciprocal Y_0 is introduced in order to make C_{em} and C_{me} have the dimensions of meters^{-3}. The total y-directed electric interaction field is E_i, while H_i is the total z-directed magnetic interaction field. The effective field acting to polarize each disk is the sum of the incident field plus the total interaction field.

The electric dipole moment induced in the disk in the y direction is given by

$$\begin{aligned} P &= \alpha_e\epsilon_0[E_{\text{inc}} + E_{ie} + E_{im}] \\ &= \alpha_e\epsilon_0E_{\text{inc}} + \alpha_eC_{ee}P + \alpha_eC_{em}(\mu_0\epsilon_0)^{1/2}M. \end{aligned} \tag{4a}$$

Similarly, the magnetic dipole moment induced in the z direction is found to be

$$M = \alpha_mH_{\text{inc}} + \alpha_mC_{mm}M + \alpha_mC_{me}(\mu_0\epsilon_0)^{-1/2}P, \tag{4b}$$

where α_e and α_m are the electric and magnetic polarizabilities of the disk, respectively. Solving for P and M gives

$$P = \frac{(1-\alpha_mC_{mm})\alpha_e\epsilon_0E_{\text{inc}} + \alpha_e\alpha_m\epsilon_0C_{em}Z_0H_{\text{inc}}}{(1-\alpha_eC_{ee})(1-\alpha_mC_{mm}) - \alpha_e\alpha_mC_{em}C_{me}}, \tag{5a}$$

$$M = \frac{(1-\alpha_eC_{ee})\alpha_mH_{\text{inc}} + \alpha_e\alpha_mC_{me}Y_0E_{\text{inc}}}{(1-\alpha_eC_{ee})(1-\alpha_mC_{mm}) - \alpha_e\alpha_mC_{em}C_{me}}. \tag{5b}$$

In practice, the terms α_eC_{em}, α_mC_{me}, and $\alpha_e\alpha_m$ are small so that (5a) and (5b) reduce approximately to the more familiar expressions

$$P = \frac{\alpha_e\epsilon_0E_{\text{inc}}}{1-\alpha_eC_{ee}}, \tag{6a}$$

$$M = \frac{\alpha_mH_{\text{inc}}}{1-\alpha_mC_{mm}}. \tag{6b}$$

These results are equivalent to a neglect of the interaction between the electric and magnetic dipoles. The analysis to follow will give expressions for the interaction constants C_{ee}, C_{em}, C_{me} and C_{mm} [see (28), (31), and (40)].

Derivation of Interaction Constants

Consider an infinite two-dimensional array of magnetic dipoles $M\mathbf{a}_z$ with a relative phase e^{-jhma} along the x axis. From symmetry considerations, the scattered field is such that conducting planes can be inserted into the lattice at $y=\pm b/2$ as in Fig. 2.

The field scattered by a single z-directed magnetic dipole, located at the origin, will be determined first. This field may be found from a magnetic Hertzian potential Π_z' as follows:

$$\boldsymbol{E} = -j\omega\mu_0\nabla\times\boldsymbol{a}_z\Pi_z', \tag{7a}$$

$$\boldsymbol{H} = k_0^2\Pi_z'\boldsymbol{a}_z + \nabla\nabla\cdot\boldsymbol{a}_z\Pi_z', \tag{7b}$$

where

$$\nabla^2\Pi_z' + k_0^2\Pi_z' = -M\delta(x)\delta(y)\delta(z), \tag{8}$$

and $\delta(x)$, etc., is the unit impulse function. Since E_z must vanish at $y=\pm b/2$, a suitable form for Π_z' is

$$\Pi_z' = \sum_{n=0}^{\infty} f_n(r)\cos 2n\pi y/b, \tag{9}$$

where $f_n(r)$ is a suitable function of $r=(x^2+z^2)^{1/2}$ to be determined. Substituting (9) into (8), multiplying both sides by $\cos 2n\pi y/b$, and integrating over $-b/2\le y\le b/2$ gives

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f_n}{\partial r} + \left[k_0^2 - \left(\frac{2n\pi}{b}\right)^2\right]f_n = -\frac{\epsilon_{0n}}{b}M\delta(r), \tag{10}$$

where $\delta(r)=\delta(x)\delta(z)$ and $\epsilon_{0n}=1$; $n=0$ and $\epsilon_{0n}=2$ for $n>0$. The solution to (10) which is bounded as $r\to\infty$ is

[4] Z. A. Kaprielian, "Anisotropic effects in geometrically isotropic lattices," *J. Appl. Phys.*, vol. 29, pp. 1052–1063; July, 1958.

$f_n(r) = a_n K_0(\gamma_n r)$ where K_0 is the modified Bessel function of the second kind, and $\gamma_n^2 = (2n\pi/b)^2 - k_0^2$. As $r \to 0$ the solution must have a logarithmic singularity of strength $-\epsilon_{0n} M/2\pi b \ln r$ and since $K_0(\gamma_n r) \to -\ln r$ as $r \to 0$ the coefficient a_n is given by

$$a_n = \frac{\epsilon_{0n} M}{2\pi b}. \tag{11}$$

Hence, the potential due to a single dipole is

$$\Pi_z' = \frac{M}{2\pi b}\left[K_0(jk_0 r) + 2\sum_{n=1}^{\infty} K_0(\gamma_n r)\cos 2n\pi y/b\right], \tag{12}$$

since $\gamma_0 = jk_0$. The Bessel function K_0 with imaginary argument is proportional to the Hankel function $H_0^2(k_0 r)$.

For dipoles at $x = ma$, $m = \pm 1, \pm 2, \cdots$, and having a relative phase $\exp(-jhma)$, the required potential is readily obtained by using (12) and is

$$\Pi_{z1} = \frac{M}{2\pi b}\left[\sum_{m=-\infty}^{\infty}{}' e^{-jhma} K_0(jk_0\sqrt{z^2 + (ma - x)^2}) \right. \tag{13}$$

$$\left. + 2\sum_{m=-\infty}^{\infty}{}' \sum_{n=1}^{\infty} e^{-jhma}\cos(2n\pi y/b) K_0(\gamma_n \sqrt{z^2 + (ma - x)^2})\right].$$

The prime means omission of the term $m = 0$. The double series converges very rapidly, since b is limited to be less than a half wavelength; hence, γ_n is real for $n > 0$ and K_0 decays rapidly. The single series will be transformed to a more rapidly converging form by an application of the Poisson summation formula.

Consider the series

$$\sum_{m=-\infty}^{\infty} S_0(ma) = \sum_{m=-\infty}^{\infty} K_0(jk_0\sqrt{z^2 + (ma)^2}). \tag{14}$$

To apply the Poisson summation formula, the following Fourier transform is required

$$\int_{-\infty}^{\infty} e^{-jwu} K_0(jk_0\sqrt{z^2 + u^2})du$$

$$= \pi \frac{\exp - |z| \sqrt{w^2 - k_0^2}}{\sqrt{w^2 - k_0^2}}. \tag{15}$$

According to the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} S_0(ma) = \frac{1}{a}\sum_{m=-\infty}^{\infty} g(2m\pi/a), \tag{16}$$

where $g(w)$ is the Fourier transform of $S_0(u)$. Thus, (14) becomes

$$\sum_{m=-\infty}^{\infty} K_0(jk_0\sqrt{z^2 + (ma)^2})$$

$$= \frac{\pi}{a}\sum_{m=-\infty}^{\infty} \frac{\exp - |z| \sqrt{(2m\pi/a)^2 - k_0^2}}{\sqrt{(2m\pi/a)^2 - k_0^2}}. \tag{17}$$

Multiplying each term in (14) by e^{-jhma} replaces $(2m\pi/a)$ by $(2m\pi/a) + h$ in the transformed series. Replacing (ma) in (14) by $(ma - x)$ is equivalent to multiplying each term in the transformed series by $e^{-j2m\pi x/a}$. With the aid of these operational formulas, it is found that

$$\sum_{m=-\infty}^{\infty} e^{-jhma} K_0[jk_0\sqrt{z^2 + (ma - x)^2}]$$

$$= e^{-jhx}\sum_{m=-\infty}^{\infty} e^{-jh(ma-x)} K_0[jk_0\sqrt{z^2 + (ma - x)^2}]$$

$$= e^{-jhx}\frac{\pi}{a}\sum_{m=-\infty}^{\infty}\frac{e^{-\Gamma_m|z| - j2m\pi x/a}}{\Gamma_m}, \tag{18}$$

where

$$\Gamma_m^2 = [(2m\pi/a) + h]^2 - k_0^2.$$

The potential arising from dipoles at $x = ma$; $m = \pm 1, \pm 2, \cdots$, may now be expressed as

$$\Pi_{z1} = \frac{M}{2\pi b}\left[\frac{\pi}{a}\sum_{m=-\infty}^{\infty}\frac{e^{-\Gamma_m|z|}e^{-j(h+2m\pi/a)x}}{\Gamma_m}\right.$$

$$- K_0(jk_0\sqrt{x^2 + z^2}) + 2\sum_{m=-\infty}^{\infty}{}'\sum_{n=1}^{\infty} e^{-jhma}$$

$$\left.\cdot\cos(2n\pi y/b) K_0(\gamma_n\sqrt{z^2 + (ma - x)^2})\right]. \tag{19}$$

To obtain the potential due to all the dipoles in the lattice, except the dipole at the origin, the potential from dipoles located at $y = \pm mb$; $m = 1, 2, \cdots$; $x = z = 0$ must be added to (19). These dipoles are the images of the dipole at the origin, and the partial potential contributed by these is

$$\Pi_{z2} = \frac{M}{4\pi}\sum_{m=-\infty}^{\infty}{}' \frac{\exp - jk_0\sqrt{x^2 + z^2 + (mb - y)^2}}{\sqrt{x^2 + z^2 + (mb - y)^2}}. \tag{20}$$

Consider next the field scattered from y-directed electric dipoles of moment P and having a relative phase e^{-jhma}. The scattered field may be obtained from a vector potential A_y' with a single y component by means of the equations

$$\boldsymbol{E} = (j\omega\mu_0\epsilon_0)^{-1}(k_0^2 \boldsymbol{a}_y A_y' + \nabla\nabla\cdot\boldsymbol{a}_y A_y'), \tag{21a}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{a}_y A_y', \tag{21b}$$

and

$$\nabla^2 A_y' + k_0^2 A_y' = -j\omega\mu_0 P\delta(x)\delta(y)\delta(z), \tag{21c}$$

for a single dipole at the origin. The boundary conditions on A_y' are the same as those for Π_z', and (21c) is similar to (8). Therefore, the solution for the total vector potential A_y from all dipoles except the dipole at the origin is the same as the solution for $\Pi_{z1} + \Pi_{z2}$ but with M replaced by $j\omega\mu_0 P$. Thus,

$$A_y = \frac{j\omega\mu_0 P}{2\pi b}\left[\frac{\pi}{a}\sum_{m=-\infty}^{\infty}\frac{e^{-\Gamma_m|z|}e^{-j(h+2m\pi/a)x}}{\Gamma_m}\right.$$

$$- K_0(jk_0\sqrt{x^2+z^2}) + 2\sum_{m=-\infty}^{\infty}{}'\sum_{n=1}^{\infty} e^{-jhma}$$

$$\left.\cdot\cos(2n\pi y/b)K_0(\gamma_n\sqrt{z^2+(ma-x)^2})\right]$$

$$+\frac{j\omega\mu_0 P}{4\pi}\left[\sum_{m=-\infty}^{\infty}{}'\frac{\exp - jk_0\sqrt{x^2+z^2+(mb-y)^2}}{\sqrt{x^2+z^2+(mb-y)^2}}\right]. \quad (22)$$

The y-directed interaction field E_{ie} is given by

$$E_{ie} = (j\omega\mu_0\epsilon_0)^{-1}\left(k_0^2 + \frac{\partial^2}{\partial y^2}\right)A_y. \quad (23)$$

Using (22) and carrying out the operations indicated in (23), placing $x=y=0$, and letting z tend to zero, it is found that

$$E_{ie} = \lim_{z\to 0}\frac{k_0^2 P}{2\pi\epsilon_0 b}\left[\frac{\pi}{a}\sum_{m=-\infty}^{\infty}\frac{e^{-\Gamma_m|z|}}{\Gamma_m} - K_0(jk_0|z|)\right]$$

$$-\frac{2P}{\pi\epsilon_0 b}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\cos(hma)\gamma_n^2 K_0(\gamma_n ma)$$

$$+\frac{P}{4\pi\epsilon_0}\left[4jk_0\sum_{m=1}^{\infty}\frac{e^{-jk_0mb}}{(mb)^2} + 4\sum_{m=1}^{\infty}\frac{e^{-jk_0mb}}{(mb)^3}\right]. \quad (24)$$

Now

$$\Gamma_m^{-1} = \frac{a}{2|m|\pi}\left[1+\frac{ha}{m\pi}-\frac{k_0^2-h^2}{(2m\pi)^2}a^2\right]^{-1/2}$$

$$= \frac{a}{2|m|\pi} + 0\left[\frac{1}{m^2}\right]. \quad (25)$$

Thus, the dominant part of the series

$$\frac{\pi}{a}\sum_{m=-\infty}^{\infty}{}'\frac{e^{-\Gamma_m|z|}}{\Gamma_m}$$

is

$$\frac{\pi}{a}\sum_{m=-\infty}^{\infty}{}'\frac{e^{-2|m|\pi|z|/a}}{2|m|\pi/a} = \sum_{m=1}^{\infty}\frac{e^{-2\pi|z|m/a}}{m}.$$

This series is readily summed by standard methods to give[5]

$$\sum_{m=1}^{\infty}\frac{e^{-2\pi m|z|/a}}{m} = -\ln 2\sinh\frac{\pi|z|}{a} + \frac{\pi|z|}{a}$$

$$\to -\ln 2\pi|z|/a \text{ as } z\to 0. \quad (26)$$

Also,

$$\lim_{z\to 0}K_0(jk_0|z|) = -(\gamma + \ln jk_0|z|/2)$$

$$= -\gamma - j\pi/2 - \ln k_0|z|/2, \quad (27)$$

[5] R. E. Collin, "Field Theory of Guided Waves," McGraw-Hill Book Co., Inc., New York, N. Y.; 1960. See especially Sec. A-6.

where $\gamma = 0.577$ is Euler's constant. Thus, the logarithmic singularity due to the Bessel function $K_0(jk_0|z|)$ is cancelled by the logarithmic singularity arising from the dominant part of the series, *i.e.* from (26).

In (24), the first series may be written as a dominant series and a rapidly converging series. The series over $(mb)^{-2}$ and $(mb)^{-3}$ are readily summed. After summing these series and making use of (26) and (27), the following final result is obtained:

$$E_{ie} = \frac{k_0^2 P}{2ab\epsilon_0}\left[-\frac{a}{\pi}\left(\ln\frac{4\pi}{k_0 a}-\gamma\right)+j\left(\frac{a}{2}-\frac{1}{\sqrt{k_0^2-h^2}}\right)\right.$$

$$\left.+\sum_{m=1}^{\infty}\left(\frac{1}{\Gamma_m}+\frac{1}{\Gamma_{-m}}-\frac{a}{m\pi}\right)\right]$$

$$-\frac{2P}{\pi b\epsilon_0}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\gamma_n^2\cos(hma)K_0(\gamma_n ma)$$

$$+\frac{P}{\pi\epsilon_0 b^3}\left[1.2 - \frac{k_0^2b^2}{2}\ln k_0 b + \frac{k_0^2b^2}{4} + \frac{k_0^4b^4}{96}\right.$$

$$\left.-j\left(\frac{\pi}{4}k_0^2b^2 - \frac{k_0^3b^3}{6}\right)\right] = C_{ee}\frac{P}{\epsilon_0}\cdot \quad (28)$$

This equation determines the dynamic interaction constant C_{ee}. The double series involving K_0 converges very rapidly.

Determination of the interaction constant C_{me} requires evaluation of the interaction field H_{ie} at the origin. This interaction field is given by

$$H_{ie} = \mu_0^{-1}\frac{\partial A_y}{\partial x}$$

$$= \frac{j\omega P}{2\pi b}\left[-j\frac{\pi}{a}\sum_{m=-\infty}^{\infty}e^{-\Gamma_m|z|}e^{-j(h+2m\pi/a)x}\frac{h+2m\pi/a}{\Gamma_m}\right.$$

$$+\frac{jk_0x}{r}K_1(jk_0r) + 2\sum_{m=-\infty}^{\infty}{}'\sum_{n=1}^{\infty}$$

$$\left.\cdot e^{-jhma}\frac{\gamma_n|(ma-x)|}{\sqrt{z^2+(ma-x)^2}}K_1(\gamma_n\sqrt{z^2+(ma-x)^2})\right]$$

$$+\frac{j\omega P}{4\pi}\sum_{m=-\infty}^{\infty}{}'\left[-jk_0\frac{e^{-jk_0r_m}}{r_m}-\frac{e^{-jk_0r_m}}{r_m^2}\right]\frac{x}{r_m}, \quad (29)$$

where $r^2=x^2+z^2$, $r_m^2=z^2+x^2+(mb-y)^2$. As x and z tend to zero,

$$H_{ie} = \lim_{z\to 0}\frac{j\omega P}{2\pi b}\left[-j\frac{\pi}{a}\sum_{m=-\infty}^{\infty}e^{-\Gamma_m|z|}\frac{h+2m\pi/a}{\Gamma_m}\right.$$

$$\left.-4j\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(hma)\gamma_n K_1(\gamma_n ma)\right]. \quad (30)$$

Now,

$$\Gamma_m^{-1} = \frac{a}{2|m|\pi}\left(1-\frac{ha}{2m\pi}\right)+0\left(\frac{1}{m^3}\right);$$

and hence the dominant part of the first series in (30) is

$$\sum_{m=-\infty}^{\infty}{}' e^{-2|m\pi z|/a} \frac{2m\pi}{a}\left(\frac{a}{2|m|\pi} - \frac{ha^2 \operatorname{sg} m}{(2m\pi)^2}\right) + \sum_{m=-\infty}^{\infty}{}' e^{-2|m\pi z|/a} \frac{ha}{2|m|\pi},$$

while the correction series is

$$\sum_{m=-\infty}^{\infty}\left[\frac{2m\pi}{\Gamma_m a} - \frac{2m\pi}{a}\left(\frac{a}{2|m|\pi} - \frac{ha^2 \operatorname{sg} m}{(2m\pi)^2}\right)\right] + \sum_{m=-\infty}^{\infty}{}'\left[\frac{h}{\Gamma_m} - \frac{ha}{2|m|\pi}\right],$$

where $\operatorname{sg} m = 1$ for $m > 0$ and -1 for $m < 0$. The dominant part of the series vanishes, since the terms are odd functions of m. Thus, only the correction series and the double series in (30) contribute, and the final result for H_{ie} is

$$H_{ie} = \frac{\omega P}{2\pi b}\left[-j\frac{\pi}{a}\frac{h}{\sqrt{k_0^2 - h^2}} + \frac{\pi}{a}\sum_{m=1}^{\infty}\left(\frac{h + 2m\pi/a}{\Gamma_m} + \frac{h - 2m\pi/a}{\Gamma_{-m}}\right) + 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\gamma_n \sin(hma)K_1(\gamma_n ma)\right] = \frac{C_{me}\omega P}{k_0}. \quad (31)$$

The double series converges very rapidly and only one or two terms is usually required. From (2b) it is seen that $H_{ie} = C_{me} Y_0 P/\epsilon_0$ and hence the right hand side of (31) when divided by $Y_0 P/\epsilon_0$ gives the interaction constant C_{me}.

The y-directed electric interaction field E_{im} due to the magnetic dipoles may be obtained from (31) by replacing P by $\mu_0 M$. When this is done and (2a) is used, it is found that

$$C_{em} = C_{me}. \quad (32)$$

As a final step, the interaction constant C_{mm} must be found. The interaction field H_{im} is given by

$$H_{im} = \left(k_0^2 + \frac{\partial^2}{\partial z^2}\right)\Pi_z, \quad x = y = z = 0, \quad (33)$$

where $\Pi_z = \Pi_{z1} + \Pi_{z2}$ and Π_{z1} is given by (19) and Π_{z2} by (20). The second term in (33) gives

$$\left.\frac{\partial^2 \Pi_z}{\partial z^2}\right|_0 = \lim_{z\to 0}\frac{M}{2\pi b}\left[\frac{\pi}{a}\sum_{m=-\infty}^{\infty}\Gamma_m e^{-\Gamma_m|z|} + k_0^2\left(K_0(jk_0|z|) + \frac{K_1(jk_0|z|)}{jk_0|z|}\right)\right] - \frac{M}{\pi b}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}{}'\frac{\gamma_n}{|m|a}e^{-jhma}K_1(\gamma_n ma) - \frac{M}{2\pi}\sum_{m=1}^{\infty}\left[\frac{jk_0 e^{-jk_0 mb}}{(mb)^2} + \frac{e^{-jk_0 mb}}{(mb)^3}\right]. \quad (34)$$

In order to evaluate the limiting value of this expression, the dominant part of the first series must be found first. Since

$$\Gamma_m = \frac{2|m|\pi}{a} + h \operatorname{sg} m - \frac{k_0^2 a}{4|m|\pi} + 0\left(\frac{1}{m^2}\right),$$

the series

$$\frac{\pi}{a}\sum_{m=-\infty}^{\infty}{}'\Gamma_m e^{-\Gamma_m z}$$

may be written as a dominant series,

$$\frac{\pi}{a}\sum_{m=-\infty}^{\infty}{}' e^{-2|mz|\pi/a}\left[\frac{2|m|\pi}{a} + h\operatorname{sg} m - \frac{k_0^2 a}{4|m|\pi}\right],$$

plus a correction series

$$\frac{\pi}{a}\Gamma_0 + \frac{\pi}{a}\sum_{m=-\infty}^{\infty}{}'\left[\Gamma_m - \frac{2|m|\pi}{a} - h\operatorname{sg} m + \frac{k_0^2 a}{4|m|\pi}\right] = j\frac{\pi}{a}\sqrt{k_0^2 - h^2} + \frac{\pi}{a}\sum_{m=1}^{\infty}\left[\Gamma_m + \Gamma_{-m} - \frac{4m\pi}{a} + \frac{k_0^2 a}{2m\pi}\right]. \quad (35)$$

The dominant part of the series sums to

$$\left(\frac{2\pi}{a}\right)^2\sum_{m=1}^{\infty} m e^{-2m\pi|z|/a} - \frac{k_0^2}{2}\sum_{m=1}^{\infty}\frac{e^{-2m\pi|z|/a}}{m}$$

$$= \left(\frac{2\pi}{a}\right)^2 \frac{1}{4\sinh^2 \pi|z|/a} + \frac{k_0^2}{2}\left[\ln 2\sinh\frac{\pi|z|}{a} - \frac{\pi|z|}{a}\right]$$

$$\to \frac{1}{|z|^2} - \frac{\pi^2}{3a^2} + \frac{k_0^2}{2}\ln 2\pi|z|/a, \quad (36)$$

since

$$(\sinh^2 \pi|z|/a)^{-1} \to \frac{a^2}{\pi^2|z|^2} - \frac{1}{3} \text{ as } z \to 0.$$

The limiting form of the Bessel function term in the first part of (34) is

$$k_0^2\left[K_0(jk_0|z|) + \frac{K_1(jk_0|z|)}{jk_0|z|}\right] \to k_0^2\left[-\frac{\gamma}{2} - \frac{1}{2}\ln\frac{jk_0|z|}{2} - \frac{1}{k_0^2|z|^2} - \frac{1}{4}\right],$$

as $z \to 0$.

By adding the above to the series term (35) and (36), the singular terms cancel and one obtains the result

$$\frac{M}{2\pi b}\left[j\frac{\pi}{a}\sqrt{k_0^2 - h^2} - \frac{\pi^2}{3a^2} + \frac{k_0^2}{2}\ln\frac{4\pi}{k_0 a} - \frac{\gamma k_0^2}{2} - j\frac{k_0^2 \pi}{4} - \frac{k_0^2}{4} + \frac{\pi}{a}\sum_{m=1}^{\infty}\left(\Gamma_m + \Gamma_{-m} - \frac{4m\pi}{a} + \frac{k_0^2 a}{2m\pi}\right)\right], \quad (37)$$

after multiplying by $M/2\pi b$.

The remaining part of (34) may be summed to give

$$-\frac{2M}{\pi b}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\gamma_n}{ma}\cos(hma)K_1(\gamma_n ma)$$

$$-\frac{M}{2\pi b^3}\left[1.2-\frac{k_0^2b^2}{2}\ln k_0 b+\frac{k_0^2b^2}{4}+\frac{k_0^4b^4}{96}\right.$$

$$\left.-j\left(\frac{\pi}{4}k_0^2b^2-\frac{k_0^3b^3}{6}\right)\right]. \quad (38)$$

The only part left to be evaluated to obtain H_{im} is the term $k_0^2\Pi_z$. The evaluation of this term is similar to the evaluation of $k_0^2A_y$ given earlier. The final result for $x=y=z=0$, is

$$k_0^2\Pi_z=\frac{k_0^2M}{2\pi b}\left[-\ln\frac{4\pi}{k_0a}+\gamma+j\frac{\pi}{2}-j\frac{\pi}{a\sqrt{k_0^2-h^2}}\right.$$

$$+\frac{\pi}{a}\sum_{m=1}^{\infty}\left(\frac{1}{\Gamma_m}+\frac{1}{\Gamma_{-m}}-\frac{a}{m\pi}\right)$$

$$\left.+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\cos(hma)K_0(\gamma_n ma)\right]$$

$$-\frac{k_0^2M}{2\pi b}\left[-j\frac{k_0b}{2}+j\frac{\pi}{2}+\ln 2\sin k_0b/2\right]. \quad (39)$$

The field H_{im} is given by the sum of (37), (38) and (39) and is equal to $C_{mm}M$. Hence, the interaction constant C_{mm} is given by

$$C_{mm}=-\frac{1}{2\pi b}\left\{\left[\frac{1.2}{b^2}+\frac{\pi^2}{3a^2}+\frac{k_0^2}{2}(1-\gamma)+\frac{k_0^4b^2}{96}\right.\right.$$

$$\left.+\frac{k_0^2}{2}\ln\frac{8\pi}{k_0^2ab}(1-\cos k_0b)\right]$$

$$-\frac{k_0^2\pi}{a}\sum_{m=1}^{\infty}\left[\frac{1}{\Gamma_m}+\frac{1}{\Gamma_{-m}}+\frac{\Gamma_m+\Gamma_{-m}}{k_0^2}-\frac{a}{2m\pi}-\frac{4m\pi}{k_0^2a}\right]$$

$$-4k_0^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\cos(hma)\left[K_0(\gamma_n ma)-\frac{\gamma_n}{mk_0^2a}K_1(\gamma_n ma)\right]$$

$$\left.-j\left[\frac{k_0^3b}{3}-\frac{\pi h^2}{a(k_0^2-h^2)^{1/2}}\right]\right\}. \quad (40)$$

This completes the derivation of the dynamic field interaction constants.

When the lattice spacings a and b are small compared with the wavelength λ_0, static field interaction constants may be used. These may be obtained by placing k_0^2 equal to zero in the expressions for C_{ee}, C_{em}, C_{me} and C_{mm}. It is readily found that

$$C_{ee}'=\frac{1.2}{\pi b^3}-\frac{8\pi}{b^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}n^2K_0(2nm\pi a/b), \quad (41a)$$

$$C_{em}'=C_{me}'=0, \quad (41b)$$

$$C_{mm}'=-\frac{0.6}{\pi b^3}-\frac{\pi}{6a^2b}$$

$$-\frac{4}{ab^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{m}K_1(2nm\pi a/b), \quad (41c)$$

where the prime denotes static interaction constants. The static-field case C_{mm}' should be symmetrical in the variables a and b because of the symmetry involved in the two-dimensional lattice structure. Although (41c) seems to violate this condition, it may be shown that C_{mm}' as given by (41c) is also equal to[6]

$$C_{mm}'=-\frac{1.2}{\pi b^3}-\frac{1.2}{\pi a^3}+8\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$$

$$\cdot\left[\frac{m^2}{b^3}K_0(2nm\pi a/b)+\frac{m^2}{a^3}K_0(2nm\pi b/a)\right]. \quad (42)$$

Conclusions

Suitable methods for evaluating the dynamic interaction fields in a two-dimensional lattice have been given. The final results for a particular case have been presented in terms of a set of interaction constants whose numerical values are readily computed. Reference to (5) shows that interaction fields will be of importance whenever the product of the element polarizability and the appropriate interaction constant is not negligible, compared to unity. This usually implies elements (such as disks and apertures) which are an appreciable fraction of a wavelength in size. For such elements, the polarizabilities are not, in general, given very accurately by the static formulas. Full advantage in the use of the dynamic interaction field analysis presented here is therefore limited to those elements for which higher order approximations to the polarizabilities are available. For the circular disk and aperture, the polarizabilities α_e and α_m have been evaluated up to and including terms in $(k_0r)^2$ where r is the disk radius.[7] Application of the results presented here will be discussed in a future paper.

[6] R. E. Collin and W. Eggimann, "Evlauation of Dynamic Interaction Fields in a Two Dimensional Lattice," Case Inst. Tech., Cleveland, Ohio, Sci. Rept. No. 12, issued under Contract AF 19(604)3887, March, 1960.

[7] W. H. Eggimann, "Higher order evaluation of dipole moments of a small circular disk," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, (Correspondence), p. 573; September, 1960.